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DYONIC BLACK HOLES AND RELATED SOLITONS

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Abstract. There is a growing literature on dyonic black holes as they appear in string theory. Here we examine the correspondence limit of a dyonic black hole which is not supersymmetric. Assuming the existence of a dyon with non-supersymmetric Kerr-Schild structure, we calculate its gravitational and electromagnetic fields and compute its mass and angular momentum to obtain a modified B.P.S. relation. The contribution of the angular momentum to the mass appears in the condition for the appearance of a horizon. One of the advantages of the Kerr-Schild frame is the possibility of a Lorentz covariant treatment since gravitational pseudo-energy-momentum tensor vanishes in this frame.

The solutions coming from string theory exhibit a central singularity. We briefly discuss the possibility that there are true solitonic solutions free of all singularities. We would expect these solitons to show nodding radial behavior in contrast to the known stringy black holes.

1. Introduction.

At the level of classical field theory and special relativity, theoretical models of the elementary particles have infinite mass unless they are solitonic. However, dyonic solitons do appear naturally in particular non-Abelian field theories¹ and at the level of special relativity satisfy the B.P.S. relation² connecting mass, electric and magnetic charge:

$$m^2 \geq e^2 + g^2 \tag{1.1}$$

There is a similar bound that has been established at the general relativistic level, namely³

$$m^2 \geq G^{-1}(e^2 + g^2) . \tag{1.2}$$

The extension to general relativity is obviously necessary since a satisfactory description of elementary particles must contain gravitational couplings, and a natural candidate for an elementary particle is possibly a solitonic version of a black hole. In recent work there have been many attempts to understand these putative particles, including spinning black holes, as they appear in higher dimensional and locally supersymmetric theories.⁴ This work has also led to interesting conjectures about the Bekenstein-Hawking entropy of black holes.

The particle-like solutions of the field equations, the so-called solitons, coming from string theories appear always to exhibit central singularities. In this respect they resemble Schwarzschild black holes and differ from the original idea of a soliton as a classical lump of field with no singularities. We are here concerned mainly with the black-hole type of particle but we shall also briefly consider the possibility of singularity free solitons.

We shall study the rotating dyon at the general relativistic level without the complications of higher dimensionality and local supersymmetry. Our speculative input will be confined to the assumptions that dyons⁵ do exist and may be described by a Kerr-Schild structure.⁶ We should also like to compare the mass of this specific structure with that predicted by the general relations (1.1) and (1.2).

One of the advantages of the Kerr-Schild representation of a spinning source is the possibility of a Lorentz covariant treatment⁷ since the gravitational pseudo-energy-momentum tensor (p.e.m.t.) vanishes in this representation.⁸ Passing from a general coordinate system to Kerr-Schild coordinates therefore cancels the gravitational energy and momentum and may be interpreted as a kind of acceleration according to the equivalence principle. Additional Poincaré transformations will not change the Kerr-Schild metric. There are also

linear but complex translations which lead from the neutral spinning source to either the Schwarzschild source⁹ or to the charged spinning source.¹⁰ Here we shall use the method of complex translation to obtain a description of a 4-dimensional dyon.

An important role in the considerations of this paper is played by the gravitational energy. Since gravitational energy is not localizable, there is an arbitrariness in discussing it and consequently there have been many different proposals for the total energy-momentum of an isolated system.¹¹ These different expressions for the pseudo-energy-momentum tensor all lead to energy-momentum vectors that may be written as essentially equivalent surface integrals. The problem has been discussed in generality by Arnowitt, Deser, and Misner.¹² Our problem is simpler since we are assuming not only the Kerr-Schild metric but also time independence. We shall show that in this metric the contribution of the gravitational field to the pseudo-energy-momentum tensor vanishes exactly if the source field is conformal (traceless).

If one takes the view that string theory is essentially correct, the first part of this paper may be regarded as the correspondence limit of a higher dimensional construction such as M theory.

The second part of this paper distinguishes between string solitons and “true” solitons as candidates for elementary particles.

2. The General Relativistic Structure of a Rotating Dyon.

Since the dyon is the source of both an electric (e) and magnetic (g) charge, it is also the source of two independent fields, $F_{\mu\nu}^{(e)}$ and $F_{\mu\nu}^{(g)}$ with associated vector potentials $A_\mu^{(e)}$ and $A_\mu^{(g)}$ as well as energy-momentum tensors $\theta_{\mu\nu}^{(e)}$ and $\theta_{\mu\nu}^{(g)}$. For $\theta_{\mu\nu}^{(A)}$ we have the usual construction

$$\theta_{\mu\nu}^{(A)} = \left(F_\mu^\sigma F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right)^A \quad A = (e, g) \quad (2.1)$$

and the complete energy momentum tensor of the electromagnetic field is

$$\theta_{\mu\nu} = \theta_{\mu\nu}^{(e)} + \theta_{\mu\nu}^{(g)} . \quad (2.2)$$

We do not assume that these fields are generated by a non-Abelian theory.

We are also assuming that this dyon is the rotating source of a gravitational field, $g_{\mu\nu}$, which may be written in the Kerr-Schild form:⁶

$$g_{\mu\nu} = \eta_{\mu\nu} - 2m\ell_\mu\ell_\nu \quad (2.3)$$

where $\eta_{\mu\nu}$ is the Minkowski metric (1,-1,-1,-1) and where the null vector ℓ_μ is

$$\ell_\mu = (\ell_o, \ell_o\lambda_k) \quad (2.4a)$$

$$\lambda_i\lambda_i = 1 . \quad (2.4b)$$

We shall show that⁷

$$\ell_o^2 = \left(1 - \frac{e^2 + g^2}{2m\rho} \right) \alpha(\rho) \quad (2.5a)$$

where α is the real part of a harmonic function:

$$\gamma = \alpha + i\beta \quad (2.5b)$$

and where

$$\rho = \frac{\alpha}{\alpha^2 + \beta^2} . \quad (2.5c)$$

Thus ℓ_o^2 , and therefore $g_{\mu\nu}$, is entirely fixed by the harmonic function γ .

In the uncharged case $e = g = 0$ and

$$\ell_o^2 = \alpha . \quad (2.6)$$

In this case ℓ_o^2 may be regarded as a generalization of the Newtonian potential, while β , the imaginary part of γ , is proportional to the specific angular momentum of the source.

Instead of describing ℓ_μ in terms of γ we may describe it in terms of its reciprocal, ω , which may be expressed as a complexified radial coordinate:

$$\omega = [x^2 + y^2 + (z - ia)^2]^{1/2} \quad (2.7)$$

$$= \rho + i\sigma . \quad (2.8)$$

Then ρ may be regarded as a new coordinate substituting for the usual radial coordinate, r , and σ as a new coordinate substituting for the azimuthal variable:

$$\cos \theta = \frac{z}{\rho} = -\frac{\sigma}{a} \quad (2.9)$$

$$\rho^2 - \sigma^2 = r^2 - a^2 . \quad (2.10)$$

Later we shall verify that the imaginary displacement, a , in (2.7) measures the specific angular momentum. In order to establish Eq. (2.5) we must satisfy the simultaneous field equations

$$R_{\mu\nu} = K(\theta_{\mu\nu}^e + \theta_{\mu\nu}^g) \quad (2.11)$$

$$\partial_\nu F^{A\mu\nu} = J^{A\mu} , \quad A = e, g \quad (2.12)$$

Here $K = \frac{8\pi}{c^2}k$ where k is Newton's constant and where

$$\begin{aligned} J_\mu^e &= (e, \vec{0})\delta(\vec{x}) \\ J_\mu^g &= (g, \vec{0})\delta(\vec{x}) . \end{aligned} \quad (2.13)$$

The Kerr-Schild metric has the property that the Lorentzian metric ($\eta^{\mu\nu}$), as well as $g^{\mu\nu}$, may be used to raise the indices of $F_{\mu\nu}$ and therefore Eq. (2.12) has the familiar Minkowskian solution.

The possibility of obtaining Minkowskian solutions here is one example of the use of Lorentz covariant relations to discuss the Kerr-Newman geometry. It was noted by Gürses and Gürsey⁸ that the pseudotensor $\hat{\tau}_\nu^\mu$, coupling the gravitational field to itself, vanishes in the Kerr-Schild metric if the null vector ℓ_μ is also geodesic. As a consequence, there is the following linear version of the Gupta equation:

$$\partial_\alpha \partial_\beta [\eta^{\alpha\beta} g^{\mu\nu} - \eta^{\mu\alpha} g^{\nu\beta} - \eta^{\nu\alpha} g^{\mu\beta} + \eta^{\mu\nu} g^{\alpha\beta}] = 2K\eta^{\mu\lambda}\theta_\lambda^\nu \quad (2.14)$$

where θ'_λ is the energy-momentum tensor of the non-gravitational source. Here we shall show that $\hat{\tau}^\mu_\nu$ vanishes even if ℓ_μ is not geodesic, provided that θ^μ_ν is traceless.

The solution of (2.12) is

$$F^{(A)} = \partial_\mu A^{(A)}_\nu - \partial_\nu A^{(A)}_\mu, \quad A = e, g \quad (2.15)$$

$$A^{(e)}_o = e\alpha \quad (2.16)$$

$$\vec{A}^{(e)} = \vec{\mu}^{(e)} \times \vec{\nabla}\varphi, \quad \vec{\mu}^e = (0, 0, ea) \quad (2.17)$$

$$A^{(g)}_o = g\alpha \quad (2.18)$$

$$\vec{A}^{(g)} = \vec{\mu}^{(g)} \times \vec{\nabla}\varphi, \quad \vec{\mu}^g = (0, 0, ga) \quad (2.19)$$

where $(\vec{\mu}^e, \vec{\mu}^g)$ are the dipole moments respectively associated with the electric and magnetic charges. Here α and φ are the same functions for both A^e_μ and A^g_μ , and¹²

$$\varphi = \frac{1}{a} \tan^{-1} \frac{\rho}{a}. \quad (2.20)$$

At this point both sides of Eq. (2.11) have been expressed in terms of γ as defined in (2.5). It remains only to show that the two sides agree. This step is a simple extension of the argument in Ref. 7.

3. Horizon and Bound on the Mass.

In order to describe the horizon we transform to polar coordinates

$$\begin{aligned} x + iy &= (\rho + ia)e^{i\varphi} \sin \theta \\ z &= \rho \cos \theta \\ \vec{\lambda} &= (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) . \end{aligned} \tag{3.1}$$

If $a = 0$, ρ is the usual radial coordinate and θ and φ are the usual polar angles. In that case $\vec{\lambda}$ is also a radial vector but if a does not vanish, the $\vec{\lambda}$ field defines a family of curves spiraling into the origin.

Let us also transform to new coordinates (u, v) to eliminate cross-terms in the Kerr-Schild line element. Then

$$ds^2 = E_\rho du^2 - \frac{1}{E_\rho} d\rho^2 - E_\mu dv^2 - \frac{d\mu^2}{E_\mu^2} \tag{3.2}$$

where

$$\begin{aligned} E_\rho &= \frac{1}{\rho^2} \Delta_\rho & E_\mu &= \frac{1}{\rho^2} \Delta_\mu \\ \Delta_\rho &= \rho^2 - 2m\rho + Q^2 & \Delta_\mu &= 1 - \mu^2 \\ Q^2 &= e^2 + g^2 + a^2 & \mu &= \cos \theta \end{aligned} \tag{3.3}$$

Here

$$\begin{aligned} du &= dt + [1 - (\rho^2 + a^2 \cos^2 \theta)/\Delta] d\rho - a \sin^2 \theta d\varphi \\ dv &= a dt - (\rho^2 + a^2) d\varphi . \end{aligned} \tag{3.4}$$

The horizon of the black hole is determined by

$$\Delta(\rho) = 0 . \tag{3.5}$$

Then by (3.2), at the horizon, where the red shift is infinite, $g_{uu} = 0$, $g_{\rho\rho} = \infty$. If $m^2 = Q^2$, the radius of the horizon is

$$\rho = m = Q . \tag{3.6}$$

If $m^2 < Q^2$,

$$\Delta(\rho) = (\rho - m)^2 + Q^2 - m^2 > 0 \tag{3.7}$$

and there is no horizon.

Therefore the minimum value of the mass for which there is a horizon, or the maximum value for which there is no horizon, is given by

$$m^2 = Q^2 = e^2 + g^2 + a^2 \quad (3.8)$$

where all quantities are expressed as lengths. Then the condition for the existence of a classical black hole is in general

$$m^2 \geq e^2 + g^2 + a^2 . \quad (3.9)$$

If $e = g = a = 0$, one sees that there is always a Schwarzschild horizon.

This condition may be compared with the Bogomolny relation

$$m^2 \geq e^2 + g^2 . \quad (3.10)$$

In (3.9) there is, as one would expect, an additional contribution from the energy of rotation since a is proportional to the angular momentum.

Simple duality is built into the metric (3.2) since electric and magnetic charges appear only in the combination $e^2 + g^2$. The Reissner-Nordstrom metric, $g = a = 0$, may be obtained by setting $Q = e$ and $u = t$, $\rho = r$, $v = -r^2 d\varphi$.

One may see that the parameter m appearing in the line element is the Newtonian mass. By (2.3), (2.5a)

$$\begin{aligned} g_{oo} &= \eta_{oo} - 2m\ell_o^2 \\ &= 1 - 2m \left(1 - \frac{e^2 + g^2}{2m\rho} \right) \alpha . \end{aligned} \quad (3.11)$$

Eq. (2.5c) may be inverted

$$\alpha = \frac{\rho}{\rho^2 + \sigma^2} . \quad (3.12)$$

Then

$$g_{oo} = 1 - 2m\alpha + \left(\frac{e^2 + g^2}{\rho} \right) \left(\frac{\rho}{\rho^2 + \sigma^2} \right) . \quad (3.13)$$

Asymptotically

$$g_{oo} \rightarrow 1 - \frac{2m}{\rho} + \frac{e^2 + g^2}{\rho^2} + \dots . \quad (3.14)$$

The coefficient of $\frac{1}{\rho}$ defines the Newtonian mass. In general one may show that the distant field,¹³ with the neglect of self-coupling of the gravitational field, is

$$g_{oo} \rightarrow 1 - \frac{2M}{r} + O\left(\frac{1}{r^3}\right) \quad (3.15)$$

where

$$M = \int \theta^{oo} d\vec{x} . \tag{3.16}$$

As we shall see there is no self-coupling of the gravitational field in the Kerr-Schild frame. Here θ^{oo} is the density of energy, the source of the gravitational field.

By (3.14) and (3.15) one would have

$$M = m .$$

For a macroscopic body, such as a star, it is not possible to calculate M by (3.16); but for a Kerr-Schild dyon, the near field is precisely given and the integral may be carried out.

4. The Einstein Tensor and the Conformal Current.

The general field equations are

$$G_{\mu\lambda} = R_{\mu\lambda} - \frac{1}{2}Rg_{\mu\lambda} = K\theta_{\mu\lambda} . \quad (4.1)$$

In the conformal case

$$\theta^\mu{}_\mu = 0 . \quad (4.2)$$

Then (4.1) becomes

$$R_{\mu\lambda} = K\theta_{\mu\lambda} . \quad (4.3)$$

The Kerr-Schild form of the metric implies

$$\Gamma^\sigma_{\sigma\mu} = 0 \quad (4.4)$$

since $\sqrt{-g} = 1$.

Then the Ricci tensor simplifies

$$R_{\mu\lambda} = \partial_\sigma \Gamma^\sigma_{\mu\lambda} - \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\lambda\alpha} . \quad (4.5)$$

It is useful to set

$$\Gamma^\sigma_{\mu\lambda} = \overset{1}{\Gamma}^\sigma_{\mu\lambda} + \overset{2}{\Gamma}^\sigma_{\mu\lambda}$$

where Γ^1 and Γ^2 , which are first and second order in m , are

$$\overset{1}{\Gamma}^\sigma_{\mu\lambda} = \frac{1}{2}\eta^{\sigma\tau}(\partial_\mu h_{\tau\lambda} + \partial_\lambda h_{\mu\tau} - \partial_\tau h_{\mu\lambda}) \quad (4.6a)$$

$$\overset{2}{\Gamma}^\sigma_{\mu\lambda} = \frac{1}{2}(2m\ell^\sigma\ell^\tau)(\partial_\mu h_{\tau\lambda} + \partial_\lambda h_{\mu\tau} - \partial_\tau h_{\mu\lambda}) \quad (4.6b)$$

with

$$h_{\mu\lambda} = g_{\mu\lambda} - \eta_{\mu\lambda} . \quad (4.7)$$

For the dyonic Kerr-Schild solution it may be shown that⁷

$$\partial\ell_\alpha = -C\ell_\alpha \quad (4.8)$$

$$\partial_\alpha\ell^\alpha = -D \quad (4.9)$$

where C and D are two scalar functions and

$$\partial = \ell^\mu\partial_\mu . \quad (4.10)$$

Then

$$\Gamma_{\mu\lambda}^{\sigma} = -4m^2 C \ell^{\sigma} \ell_{\mu} \ell_{\lambda} . \quad (4.11)$$

One may now compute

$$\partial_{\sigma} \Gamma_{\mu\lambda}^{\sigma} = \frac{1}{2} [\partial_{\mu} j_{\lambda} + \partial_{\lambda} j_{\mu} - \square h_{\mu\lambda}] + 4m^2 (2C^2 + DC - \partial C) \ell_{\mu} \ell_{\lambda} \quad (4.12)$$

where

$$j_{\lambda} = \partial^{\tau} h_{\tau\lambda} \quad (4.13)$$

$$= 2m(C + D) \ell_{\lambda} \quad (4.14)$$

and

$$\partial^{\tau} = \eta^{\tau\sigma} \partial_{\sigma} .$$

One also finds

$$\Gamma_{\mu\beta}^{\alpha} \Gamma_{\lambda\alpha}^{\beta} = 4m^2 F \ell_{\mu} \ell_{\lambda} \quad (4.15)$$

where

$$2F = 3C^2 + \partial_{\alpha} \ell^{\beta} \partial_{\beta} \ell^{\alpha} - \partial_{\alpha} \ell^{\beta} \partial^{\alpha} \ell_{\beta} \quad (4.16)$$

and by (4.5), (4.2) and (4.15)

$$R_{\mu\lambda} = \frac{1}{2} [\partial_{\mu} j_{\lambda} + \partial_{\lambda} j_{\mu} + \square h_{\mu\lambda}] + 4m^2 (2C^2 + DC - \partial C - F) \ell_{\mu} \ell_{\lambda} . \quad (4.17)$$

By (4.2) and (4.3), $R = 0$ and

$$g^{\mu\lambda} (\partial_{\mu} j_{\lambda} + \partial_{\lambda} j_{\mu} + \square h_{\mu\lambda}) = 0 \quad (4.18)$$

or

$$(\eta^{\mu\lambda} + 2m \ell^{\mu} \ell^{\lambda}) (\partial_{\mu} j_{\lambda} + \partial_{\lambda} j_{\mu} + \square h_{\mu\lambda}) = 0 . \quad (4.19)$$

Since terms of the first and second order in m separately vanish, we have

$$\eta^{\mu\lambda} (\partial_{\mu} j_{\lambda} + \partial_{\lambda} j_{\mu} + \square h_{\mu\lambda}) = 0$$

or

$$\partial_{\mu} j^{\mu} = 0 . \quad (4.20)$$

Therefore j^μ is conserved as a consequence of the conformal invariance of the source.

By (4.14) and (4.20), we have

$$\partial(C + D) = CD + D^2 . \quad (4.21)$$

We may also show

$$\partial(C - D) = CD - \partial_\alpha \ell^\beta \partial_\beta \ell^\alpha . \quad (4.22)$$

The mixed tensor $R^\mu{}_\lambda$ is much simpler:

$$R^\mu{}_\lambda = g^{\mu\sigma} R_{\sigma\lambda} = (\eta^{\mu\sigma} + 2m\ell^\mu \ell^\sigma) R_{\sigma\lambda} . \quad (4.23)$$

One finds

$$R^\mu{}_\lambda = (\overset{1}{R} + \overset{2}{R})^\mu{}_\lambda \quad (4.24)$$

where

$$\overset{1}{R}^\mu{}_\lambda = \frac{1}{2} [\partial^\mu j_\lambda + \partial_\lambda j^\mu + \eta^{\mu\sigma} \square h_{\sigma\lambda}] \quad (4.25)$$

$$\overset{2}{R}^\mu{}_\lambda = 2m^2 [\partial_\alpha \ell^\beta \partial_\beta \ell^\alpha - \partial_\alpha \ell^\beta \partial^\alpha \ell_\beta + 3C^2 - 2F] \ell^\mu \ell_\lambda \quad (4.26)$$

By (4.16)

$$\overset{2}{R}^\mu{}_\lambda = 0 . \quad (4.27)$$

Therefore the mixed Ricci tensor, which is the same as the Einstein tensor, is simply

$$R^\mu{}_\lambda = \frac{1}{2} [\partial^\mu j_\lambda + \partial_\lambda j^\mu - \eta^{\mu\sigma} \square h_{\sigma\lambda}] . \quad (4.28)$$

5. The Einstein Pseudo-Energy-Momentum Tensor.

The generally covariant conservation law, namely

$$G^{\mu\lambda}_{|\lambda} = K\theta^{\mu\lambda}_{|\lambda} = 0 \quad (5.1)$$

implies the conservation equation:

$$\partial_\mu(\hat{\theta}^\mu_\lambda + \hat{\tau}^\mu_\lambda) = 0 \quad (5.2)$$

where the circumflex indicates the corresponding tensor density (multiplication by $\sqrt{-g}$).

Here $\hat{\theta}^\mu_\lambda$ is the energy-momentum tensor that is the source of the gravitational field and $\hat{\tau}^\mu_\lambda$ is the contribution of the gravitational field itself. Since $\hat{\tau}^\mu_\lambda$ is not a tensor, it may vanish in one frame without vanishing in all frames.

Since $\sqrt{-g} = 1$, for the Kerr-Schild metric, the circumflex may be dropped.

The total (pseudo) energy-momentum tensor, including the contributions of both the source field and the gravitational field, namely

$$\Theta^\mu_\lambda = \theta^\mu_\lambda + \tau^\mu_\lambda \quad (5.3)$$

may be expressed in the Einstein form

$$\Theta^\mu_{E\lambda} = \partial_\gamma h^{\gamma\mu}_\lambda \quad (5.4)$$

where

$$h^{\gamma\mu}_\lambda = \frac{1}{K} g^{\mu\beta} \frac{\partial \Gamma}{\partial (\partial_\gamma g^{\lambda\beta})} \quad (5.5)$$

$$\Gamma = (\Gamma_\sigma \Gamma_{\alpha\beta}^\sigma - \Gamma_{\alpha\mu}^\lambda \Gamma_{\beta\lambda}^\mu) g^{\alpha\beta} \quad (5.6)$$

Using the Kerr-Schild metric one finds

$$h^{\gamma\mu}_\lambda = \frac{1}{K} g^{\mu\beta} \frac{\partial (\Gamma_{\varphi\tau}^\sigma \Gamma_{\psi\sigma}^\tau g^{\varphi\psi})}{\partial (\partial_\gamma g^{\lambda\beta})} \quad (5.7)$$

$$= \frac{1}{K} g^{\mu\beta} \Gamma_{\lambda\beta}^\gamma \quad (5.8)$$

and by (5.4)

$$\Theta^\mu_{E\lambda} = \frac{1}{2K} (\partial_\lambda j^\mu + \partial^\mu j_\lambda - \eta^{\mu\sigma} \square h_{\sigma\lambda}) \quad (5.9)$$

where the covariant and contravariant indices are related by the Lorentz metric.

By (4.28) and (5.9) one now has

$$R^\mu{}_\lambda = K \Theta^\mu{}_{E\lambda} = K(\theta^\mu{}_\lambda + \tau^\mu{}_\lambda) . \quad (5.10)$$

But

$$R^\mu{}_\lambda = K \theta^\mu{}_\lambda .$$

It follows that

$$\Theta^\mu{}_{E\lambda} = \theta^\mu{}_\lambda \quad (5.11a)$$

or

$$\tau^\mu{}_\lambda = 0 . \quad (5.11b)$$

Hence the gravitational p.e.m.t. vanishes in this metric.

This result depends only on (4.8) and (4.9) and is therefore more general than the theorem of Ref. 8 which seems to require that the null vector ℓ_μ be geodesic as well, i.e., that $C = 0$ in (4.8). As shown in Ref. 7 (4.8) and (4.9) hold for the charged (Kerr-Newman) case where C and D satisfy

$$C\ell_o = \frac{1}{2}|\gamma|^2 - \alpha\ell_o^2 \quad (5.12)$$

$$D\ell_o = \frac{1}{2}|\gamma|^2 + \alpha\ell_o^2 . \quad (5.13)$$

6. The Landau p.e.m.t.

The Landau-Lipshitz prescription for the total p.e.m.t. is

$$\Theta_L^{\mu\lambda} = \frac{1}{2K} \partial_\sigma h^{\mu\lambda\sigma} \quad (6.1)$$

where

$$h^{\mu\lambda\sigma} = \partial_\rho [g^{\mu\lambda} g^{\sigma\rho} - g^{\mu\sigma} g^{\lambda\rho}] \quad (6.2)$$

or

$$\begin{aligned} \Theta_L^{\mu\lambda} &= \frac{1}{2K} \partial_\sigma \partial_\rho [g^{\mu\lambda} g^{\sigma\rho} - g^{\mu\sigma} g^{\lambda\rho}] \\ &= \frac{1}{2K} \partial_\sigma \partial_\rho [\eta^{\mu\lambda} h^{\sigma\rho} + h^{\mu\lambda} \eta^{\sigma\rho} - \eta^{\mu\sigma} h^{\lambda\rho} - \eta^{\lambda\rho} h^{\mu\sigma}] \\ &= -\frac{1}{2K} [\partial^\mu j^\lambda + \partial^\lambda j^\mu - \square h^{\mu\lambda}] . \end{aligned} \quad (6.3)$$

This expression for $\Theta^{\mu\lambda}$ has the desired properties of symmetry and vanishing Lorentz covariant divergence:

$$\Theta_L^{\mu\lambda} = \Theta_L^{\lambda\mu} \quad (6.4)$$

$$\partial_\lambda \Theta_L^{\mu\lambda} = 0 . \quad (6.5)$$

$\Theta_L^{\mu\lambda}$ is more useful than $\Theta_E^{\mu\lambda}$ since it permits, by virtue of its symmetry, the easy calculation of a conserved angular momentum. By (6.3) the mixed tensor with respect to the Lorentz metric is

$$\overset{\circ}{\Theta}_L{}^\mu{}_\lambda = \eta_{\tau\lambda} \Theta^{\mu\tau} \quad (6.8a)$$

$$= -\frac{1}{2K} [\partial^\mu j_\lambda + \partial_\lambda j^\mu + \eta^{\mu\sigma} \square h_{\sigma\lambda}] \quad (6.8b)$$

so that the mixed Landau and Einstein tensors agree:

$$\overset{\circ}{\Theta}_L{}^\mu{}_\lambda = \Theta_E{}^\mu{}_\lambda . \quad (6.9)$$

On the other hand, if the index is lowered by the Kerr-Schild metric, rather than by the Lorentz metric, one finds

$$\Theta_L{}^\mu{}_\lambda = g_{\lambda\sigma} \Theta^\mu{}^\sigma{}_\lambda \quad (6.10a)$$

$$= \Theta_E{}^\mu{}_\lambda - (2m)^2 \frac{1}{2K} [C^2 - D^2 - \ell_\lambda \square \ell^\lambda] . \quad (6.10b)$$

Although the mixed tensors agree with respect to only the Lorentz metric, there is a modified Landau p.e.m.t. introduced in Ref. 8, which agrees as a mixed tensor with the Einstein p.e.m.t. provided that one also uses the Kerr-Schild metric, namely:

$$\Theta'^{\mu}_{L\nu} = -\frac{1}{2K}\partial_{\rho}g_{\nu\sigma}\partial_{\lambda}(g^{\sigma\mu}g^{\rho\lambda} - g^{\sigma\rho}g^{\mu\lambda}) . \quad (6.11)$$

Then the Einstein and Landau expressions reduce to the same simple form subject to (4.8), again extending the result of Ref. 9 which requires $C = 0$.

Finally

$$\partial_{\mu}\Theta^{\mu}_{\nu} = 0$$

implies

$$\square j_{\nu} = j_{\nu} . \quad (6.12)$$

7. Calculation of Mass.

Since the energy density is a perfect divergence, the total energy may be calculated as the flux through a closed surface at infinity, just as the electric charge may be found from a similar surface integral. Since the closed surface is taken at infinity, the metric may be chosen Lorentzian in the surface integral. The metric (2.3) has this property since $\ell_o^2 \rightarrow 0$. One commonly takes the closed surface to be spherical. For our purposes, however, it is more convenient to take this surface to be $\rho = \text{constant}$ instead of $r = \text{constant}$. Then we need the covariant form of Gauss' theorem:

$$\iiint_V F^s|_s dV = \iint_S F^s \lambda_s dS \quad (7.1)$$

where

$$F^s|_s = \frac{1}{\sqrt{g^{(3)}}} \partial_s \sqrt{g^{(3)}} F^s \quad (7.2)$$

$$dV = \sqrt{g^{(3)}} dx^1 dx^2 dx^3 \quad (7.3)$$

$$dS = \sqrt{g^{(2)}} d\varphi^1 d\varphi^2 \quad (7.4)$$

Here $\sqrt{g^{(3)}} = 1$ but $\sqrt{g^{(2)}}$ must be computed for an ellipsoidal surface of constant ρ .

By (3.1) we have

$$\begin{aligned} x &= (\rho \cos \varphi - a \sin \varphi) \sin \theta \\ y &= (a \cos \varphi + \rho \sin \varphi) \sin \theta \\ z &= \rho \cos \theta \end{aligned} \quad (7.5)$$

Then

$$g_{k\ell}^{(2)} = \sum_1^3 \frac{\partial x^s}{\partial \varphi^k} \frac{\partial x^s}{\partial \varphi^\ell}, \quad k, \ell = 1, 2 \quad (7.6)$$

where

$$\varphi^1 = \varphi, \quad \varphi^2 = \theta \quad (7.7)$$

and

$$\sqrt{g^{(2)}} = [(\rho^2 + a^2)(\rho^2 + \sigma^2)]^{1/2} \sin \theta. \quad (7.8)$$

By (7.4)

$$dS = [(\rho^2 + a^2)(\rho^2 + \sigma^2)]^{1/2} \sin \theta d\theta d\varphi. \quad (7.9)$$

Also

$$\begin{aligned}\sigma &= -a \cos \theta \\ d\sigma &= a \sin \theta \, d\theta .\end{aligned}\tag{7.10}$$

Then

$$dS = \frac{1}{a} [(\rho^2 + a^2)(\rho^2 + \sigma^2)]^{1/2} d\sigma d\varphi .\tag{7.11}$$

Since⁷

$$-\lambda^s \partial_s = \frac{\partial}{\partial \rho} .\tag{7.12}$$

λ^s is the inward normal to surfaces of constant ρ . Note also

$$\begin{aligned}\theta = 0 &\rightarrow \sigma = -a \\ \theta = \pi &\rightarrow \sigma = a .\end{aligned}\tag{7.13}$$

Finally by (7.1)

$$\iiint F^s_{|s} dV = \frac{2\pi}{a} (\rho^2 + a^2)^{1/2} \int_{-a}^a \lambda_s F^s(\rho, \sigma) (\rho^2 + \sigma^2)^{1/2} d\sigma .\tag{7.14}$$

This is obviously conserved since all fields are time independent. In general this expression would be conserved only if the total flux through the boundary surface vanishes.

8. Landau Mass.

We have for the energy-momentum vector

$$P^\mu = \int \Theta^{\mu o} dV . \quad (8.1)$$

The p.e.m.t. is by (6.1)

$$\Theta^{\mu\lambda} = \frac{1}{2K} \partial_\sigma h^{\mu\lambda\sigma} \quad (8.2)$$

and the mass is

$$P^o = \frac{1}{2K} \int \partial_k h^{ook} dV \quad (8.3)$$

$$= \frac{1}{2K} \int h^{ook} \lambda_k dS . \quad (8.4)$$

By (6.2)

$$h^{ook} = \eta^{oo} j^k + \eta^{k\ell} \partial_\ell h^{oo} \quad (8.5)$$

$$= 2m [-(C + D)\ell^k + \partial^k \ell_o^2] \quad (8.6)$$

by (4.14)

$$= 2m [-(\alpha^2 + \rho^2)\lambda^k + \partial^k \alpha(1 - \epsilon/\rho)] \quad (8.7)$$

by (5.12) and (5.13) and by (2.5a) where

$$\epsilon = \frac{e^2 + g^2}{2m} . \quad (8.8)$$

Then

$$\lambda_k h^{ook} = 2m \left[+\alpha^2 + \beta^2 + \partial\alpha - \frac{\epsilon}{\rho} \partial\alpha + \frac{\epsilon\alpha}{\rho^2} \partial\rho \right]$$

where

$$\partial = \lambda_k \partial^k .$$

But

$$\partial\gamma = \gamma^2 \quad (8.9)$$

$$\partial\alpha = \alpha^2 - \beta^2 \quad (8.10)$$

$$\partial\rho = -1 \quad (8.11)$$

Then

$$\lambda_k h^{ook} = 2m \left[+2\alpha^2 - \epsilon \left(\frac{\alpha}{\rho^2} + \frac{\alpha^2 - \beta^2}{\rho} \right) \right] . \quad (8.12)$$

Here

$$\begin{aligned} \frac{\alpha}{\rho^2} + \frac{\alpha^2 - \beta^2}{\rho} &= \frac{1}{\rho} \left(\frac{\alpha}{\rho} + \alpha^2 - \beta^2 \right) = \frac{1}{\rho} \left(\frac{1}{\rho^2 + \sigma^2} + \alpha^2 - \beta^2 \right) \\ &= \frac{1}{\rho} (\alpha^2 + \beta + \alpha^2 - \beta^2) \\ &= \frac{2\alpha^2}{\rho} . \end{aligned} \quad (8.13)$$

Then

$$\begin{aligned} \lambda_k h^{ook} &= 2m \left[+2\alpha^2 - \frac{2\alpha^2}{\rho} \epsilon \right] \\ &= 4m\alpha^2 \left(1 - \frac{\epsilon}{\rho} \right) . \end{aligned} \quad (8.14)$$

Hence

$$P^o = \frac{2m}{K} \left(1 - \frac{\epsilon}{\rho} \right) \int \alpha^2 dS . \quad (8.15)$$

This surface integral is

$$\begin{aligned} \int \alpha^2 dS &= \frac{1}{a} \int_0^{2\pi} \int_{-a}^a \alpha^2 [(\rho^2 + a^2)(\rho^2 + \sigma^2)]^{1/2} d\sigma d\varphi \\ &= 2\pi(\rho^2 + a^2)^{1/2} \frac{1}{a} \int_{-a}^a \frac{\rho^2}{(\rho^2 + \sigma^2)^2} (\rho^2 + \sigma^2)^{1/2} d\sigma \end{aligned} \quad (8.16)$$

$$= 4\pi . \quad (8.17)$$

Hence

$$M(\rho) = \frac{m}{K} \left(1 - \frac{e^2 + g^2}{em\rho} \right) . \quad (8.18)$$

According to this last equation

$$M(\rho) \leq 0 \quad \text{if} \quad \rho \leq \frac{e^2 + g^2}{2m} \quad (8.19)$$

$$M(\infty) = \frac{m}{K} . \quad (8.20)$$

One may interpret (8.19) and (8.20) by assigning an electromagnetic radius $(e^2 + g^2)/2m$ to this “particle” since all of the positive mass lies outside this radius. The limiting relation (8.20) may be interpreted as a statement of the equivalence principle.

One may be surprised that the angular momentum does not contribute directly to M , but it does determine M indirectly since (8.20) together with (3.8) requires

$$K^2 M^2 = e^2 + g^2 + a^2 \quad (8.21)$$

where M is the mass at which the horizon appears. If the mass M is greater than m , the radius of the horizon is given by

$$\rho_h^2 - 2m\rho_h + Q^2 = 0 \quad (8.22)$$

or

$$\rho_h^2 - 2m\rho_h + 2m\rho_\ell + a^2 = 0 \quad (8.23)$$

where the electromagnetic radius is

$$\rho_\ell = (e^2 + g^2)/2m . \quad (8.24)$$

Hence

$$2m(\rho_h - \rho_\ell) = \rho_h^2 + a^2 > 0 . \quad (8.25)$$

Therefore the electromagnetic radius is always shielded by the horizon.

9. Angular Momentum.

In terms of the Landau energy momentum tensor the angular momentum is

$$J^{\alpha\mu} = \int (x^\alpha \Theta_L^{\mu\nu} - x^\mu \Theta_L^{\alpha\nu}) dS_\nu \quad (9.1)$$

where dS_ν is an element of a 3-dimensional hypersurface. By (8.2)

$$J^{\alpha\mu} = \int (x^\alpha \partial_\sigma h_L^{\mu\nu\sigma} - x^\mu \partial_\sigma h_L^{\alpha\nu\sigma}) dS_\nu \quad (9.2)$$

where

$$dS_\nu = \frac{1}{3!} \epsilon_{\nu\alpha\beta\gamma} dx_1^\alpha dx_2^\beta dx_3^\gamma . \quad (9.3)$$

If all fields are time-independent then

$$J^{ik} = \int (x^i \partial_s h_L^{kos} - x^k \partial_s h_L^{ios}) d\vec{x} \quad (9.4)$$

$$\begin{aligned} &= \int \{ \partial_s (x^i h_L^{kos} - x^k h_L^{ios}) - (h^{koi} - h^{io k}) \} d\vec{x} \\ &= I^{ik} + II^{ik} \end{aligned} \quad (9.5)$$

where

$$I^{ik} = \int (x^i h_L^{kon} - x^k h_L^{ion}) dS_n \quad (9.6)$$

and

$$II^{ik} = \int (h_L^{io k} - h_L^{koi}) d\vec{x} . \quad (9.7)$$

Here dS_n is an element of a 2-dimensional surface.

In (9.6) and (9.7) $h_L^{io k}$ is the Landau tensor:

$$h^{iks} = \frac{1}{2K} \frac{\partial}{\partial x^t} H^{ikst} \quad (9.8)$$

where

$$H^{ikst} = g^{ik} g^{st} - g^{is} g^{kt} . \quad (9.9)$$

Here we have used the Kerr-Schild metric by setting $\sqrt{-g} = 1$. II^{ik} may be transformed to a surface integral by (9.8)

$$II^{ik} = \frac{1}{2K} \int (H^{ioks} - H^{kois}) \lambda_s dS \quad (9.10)$$

where the volume in (9.7) is bounded by a surface of constant ρ in (9.10). Since these surfaces are normal to the λ_s vector field, the integral I^{ik} may be expressed in the following way:

$$I^{ik} = \int (x^i h_L^{kos} - x^k h_L^{ios}) \lambda_s dS . \quad (9.11)$$

In (9.10) and (9.11) the element of area on the ellipsoidal (ρ) surface is dS .

The integrand of (9.10) is

$$\lambda_s (H^{ioks} - H^{kois}) = 2m(\ell^o)^2 (\lambda^i \lambda^k - \lambda^k \lambda^i) = 0 \quad (9.12)$$

where H^{ioks} is reduced by (9.9) and the Kerr-Schild metric. Then

$$II^{ik} = 0 . \quad (9.13)$$

The integral I^{ik} may be evaluated as follows:

$$\begin{aligned} h_L^{kos} &= \frac{m}{K} \left[\frac{\partial \ell^o}{\partial x^t} (\ell^k \eta^{st} - \ell^t \eta^{sk}) + \ell^o (\partial^s \ell^k - \eta^{sk} \partial_t \ell^t) \right] \\ h_L^{kos} \lambda_s &= \frac{m}{K} \left[\ell^o \frac{\partial \ell^o}{\partial x^t} (\lambda^k \lambda^t - \lambda^t \lambda^k) + \ell^o (\lambda_s \partial^s \ell^k - \lambda^k \partial_t \ell^t) \right] \end{aligned} \quad (9.14)$$

$$\begin{aligned} &= \frac{m}{K} [\ell^o (\lambda_s \partial^s \ell^k - \lambda^k \partial_t \ell^t)] \\ &= \frac{m}{K} (D - C) \ell^k \end{aligned} \quad (9.15)$$

and

$$(x^i h_L^{kos} - x^k h_L^{ios}) \lambda_s = \frac{m}{K} \ell^o (D - C) (x^s \lambda^k - x^k \lambda^s) .$$

Then

$$I^{ik} = \frac{m}{K} \int \ell_o (D - C) (x^i \lambda^k - x^k \lambda^i) dS . \quad (9.16)$$

Since the imaginary displacement is along z we consider I^{12} and compute⁷

$$x^1 \lambda^2 - x^2 \lambda^1 = -a \left(1 - \frac{\sigma^2}{a^2} \right) . \quad (9.17)$$

We also need

$$\begin{aligned} \ell^o (D - C) &= 2\alpha \ell_o^2 \\ \ell_o^2 &= \alpha (1 - \epsilon/\rho) . \end{aligned} \quad (9.18)$$

by (5.12) and (2.5a). Then

$$I^{12} = -\frac{2m}{K}a\left(1 - \frac{\epsilon}{\rho}\right)\left\{\int \alpha^2 dS - \frac{\rho^2}{a^2} \int \beta^2 dS\right\}. \quad (9.19)$$

The first integral is known from (8.17). The second integral is

$$\int \beta^2 dS = \frac{(\rho^2 + a^2)^{1/2}}{a} \int_0^{2\pi} \int_{-a}^a \beta^2 (\rho^2 + \sigma^2)^{1/2} d\sigma d\varphi \quad (9.20)$$

$$= \frac{2\pi}{a} (\rho^2 + a^2)^{1/2} \int_{-a}^a \frac{\sigma^2}{(\rho^2 + \sigma^2)^{3/2}} d\sigma \quad (9.21)$$

$$= -4\pi + \frac{2\pi}{a} (\rho^2 + a^2)^{1/2} \ln \frac{a + (\rho^2 + a^2)^{1/2}}{-a + (\rho^2 + a^2)^{1/2}}. \quad (9.22)$$

By (8.17) and (9.22) the total angular momentum is

$$J_3 = I^{12} = -\frac{4\pi m}{K} \left(1 - \frac{\epsilon}{\rho}\right) (\rho^2 + a^2)^{1/2} \left\{ \frac{2}{a} (\rho^2 + a^2)^{1/2} - \frac{\rho^2}{a^2} \ln \frac{a + (\rho^2 + a^2)^{1/2}}{-a + (\rho^2 + a^2)^{1/2}} \right\}. \quad (9.23)$$

Again

$$\begin{aligned} J_3(\rho) &\leq 0 \quad \text{if} \quad \rho \leq \frac{\ell^2 + g^2}{2m} \\ J_3(\infty) &= -\frac{2}{3} \frac{ma}{k}. \end{aligned} \quad (9.24)$$

We may regard the angular momentum as confined to the space outside of the “electromagnetic radius”.

We finally have

$$J_3/M = \frac{2a}{3}. \quad (9.25)$$

Similarly we find

$$\begin{aligned} x^1 \lambda^3 - x^3 \lambda^1 &= \lambda_2 \lambda_3 \\ x^2 \lambda^3 - x^3 \lambda^2 &= -\lambda_1 \lambda_3 \end{aligned} \quad (9.26)$$

Utilizing (9.16) and (3.1) one may show that J_1 and J_2 vanish.

10. Solitons.

The particle-like solutions so far discussed in this paper, as well as the string-derived solution, being descendants of the Schwarzschild solutions, all exhibit central singularities. Since these structures are also all time independent, the theorem of Penrose, Hawking, and Geroch does not directly apply. In any case this theorem requires certain conditions on the energy-momentum tensor that do not seem to be required by any fundamental principle.¹⁴ There is thus apparently no necessary requirement of a central singularity and there are certainly macroscopic examples in which the gravitational attraction is compensated in steady state structures without central singularities.

It is known that singularity free solitons may be constructed at the special relativistic level.^{15,16} The fields which are codetermined in these known structures remain finite with flat tangents at the origin and in general exhibit nodal behavior before vanishing at large distances. In this respect the constituent fields resemble the wave function of atomic and nuclear physics.

In looking for a replication of these or similar structures at the general relativistic level, two examples naturally come to mind and illustrate the complexity of the new situation. The first of these is formed by coupling the gravitational field to a gauge structure such as the Prasad-Somerfield soliton. The coupling is formally accomplished by replacing ∂_μ by $\nabla_\mu = \partial_\mu + \Gamma_\mu$ in the special relativistic equations. Since the Prasad-Somerfield solution itself already contains $1/r$ singularities, however, it is unlikely that the new soliton is singularity free.

As a second example we consider the simplest possibility, namely the gravitational field coupled to a non-linear scalar field. It is known that the nonlinear scalar field may be used to construct a singularity free soliton at the special relativistic level.^{15,16} We must, however, now satisfy the gravitational field equations as well.

Let the Lagrangian of the scalar field be

$$\begin{aligned} L &= T - V \\ T &= \frac{1}{2} g^{\mu\lambda} \partial_\mu \psi \partial_\lambda \psi \\ V &= f(\psi) . \end{aligned} \tag{10.1}$$

Then the energy-momentum tensor is

$$\theta_{\mu\lambda} = \frac{\partial L}{\partial g^{\mu\lambda}} - \frac{1}{2} g_{\mu\lambda} L . \tag{10.2}$$

The equation of motion of the ψ field is

$$g^{\mu\lambda}\nabla_\mu\partial_\lambda\psi + \frac{\partial f(\psi)}{\partial\psi} = 0 . \quad (10.3)$$

Since the conformal assumption may already imply a central singularity we do not make this assumption and therefore adopt the following gravitational field equations

$$R_{\mu\lambda} = K\Theta_{\mu\lambda} \quad (10.4)$$

where

$$\Theta_{\mu\lambda} = \theta_{\mu\lambda} - \frac{1}{2}\theta g_{\mu\lambda} \quad (10.5)$$

and

$$\theta \neq 0 . \quad (10.6)$$

Assume spherical symmetry and let λ_k be the unit radial vector. As the simplest ansatz let us again assume a Kerr-Schild metric. Then

$$\begin{aligned} \Theta_{oo} &= m\varphi f(\psi) - \frac{1}{2}f(\psi) \\ \Theta_{ok} &= m\varphi f(\psi)\lambda_k \\ \Theta_{jk} &= \frac{1}{2}\delta_{jk}f(\psi) + \frac{1}{2}\lambda_j\lambda_k \left[\left(\frac{d\psi}{dr} \right)^2 + 2m\varphi f(\psi) \right] . \end{aligned} \quad (10.7)$$

and

$$\begin{aligned} R_{oo} &= -m\nabla^2\varphi + 2m^2\varphi\nabla^2\varphi \\ R_{ok} &= 2m^2(\varphi\nabla^2\varphi)\lambda_k \\ R_{jk} &= \delta_{jk}2m\left(\frac{1}{r}\frac{d\varphi}{dr} + \frac{\varphi}{r^2}\right) + \lambda_j\lambda_k \left[m\left(\frac{d^2\varphi}{dr^2} - \frac{2\varphi}{r^2}\right) + 2m^2\varphi\nabla^2\varphi \right] \end{aligned} \quad (10.8)$$

where

$$\varphi = \ell_o^2 . \quad (10.9)$$

The gravitational equations of motion are now

$$-m\nabla^2\varphi + 2m^2\varphi\nabla^2\varphi = K[m\varphi f(\psi) - \frac{1}{2}f(\psi)] \quad (oo) \quad (10.10)$$

$$2m^2\varphi\nabla^2\varphi = Km\varphi f(\psi) \quad (ok) \quad (10.11)$$

$$2m\left(\frac{1}{r}\frac{d\varphi}{dr} + \frac{\varphi}{r^2}\right) = \frac{K}{2}f(\psi) \quad (kk) \quad (10.12)$$

$$m\left(\frac{d^2\varphi}{dr^2} - \frac{2\varphi}{r^2}\right) + 2m^2\varphi\nabla^2\varphi = \frac{K}{2}\left[\left(\frac{d\psi}{dr}\right)^2 + 2m\varphi f(\psi)\right] \quad (jk) \quad (10.13)$$

These equations must be satisfied simultaneously with (10.3) subject to the solitonic boundary conditions requiring that $\varphi(r)$ and $\psi(r)$ vanish at infinity and remain finite with flat tangents at $r = 0$. Although the Kerr-Schild form is versatile enough to be compatible with the energy momentum tensor of a dyonic field, there is no solitonic $\varphi(r)$ which is compatible with a solitonic solution $\psi(r)$ of (10.3) for any choice of the free function $f(\psi)$. This may be shown as follows.

The gravitational equations may be combined in the following way

$$(10.10) + (10.11) \rightarrow \nabla^2 \varphi = \frac{K}{2m} f(\psi) \quad (10.14)$$

$$(10.12) + (10.11) \rightarrow \frac{d^2 \varphi}{dr^2} = \frac{2\varphi}{r^2} \quad (10.15)$$

$$(10.13) + (10.14) \rightarrow \frac{d^2 \varphi}{dr^2} - \frac{2\varphi}{r^2} = \frac{K}{2m} \left(\frac{d\psi}{dr} \right)^2 \quad (10.16)$$

The solution of (10.15) which satisfies boundary conditions at $r = 0$ is $\varphi = ar^2$ but it blows up at ∞ . The second solution $\varphi = \frac{a}{r}$ blows up at the origin. Finally,

$$(10.15) + (10.16) \rightarrow \frac{d\psi}{dr} = 0 . \quad (10.17)$$

Equation (10.16) is obviously inconsistent with a solitonic solution of (10.3).

The failure of these simple choices is not surprising since the source field and the gravitational field are not parts of a larger structure in these examples. In a more promising approach one first introduces a unitary field generated by a postulated symmetry group. Then one part of the unitary field is recognized as the Einsteinian gravitational field while the remaining part is identified as the matter field. Such a split is made in the theory of the superstring and supergravity.

Let us therefore study, as a third example, a total field resulting from the reduction to four dimensions of a particular superstring theory. This field may be described by the following $N = 4$ supergravity action.¹⁷

$$S = \int d^4x \sqrt{-g} \left\{ R + \frac{1}{2} g^{\mu\lambda} \partial_{(\mu} \psi \partial_{\lambda)} \psi^* - f(\psi) [F_{\mu\lambda} F^{\mu\lambda} + G_{\mu\lambda} G^{\mu\lambda}] \right\} . \quad (10.18)$$

Here ψ is a complex scalar, whose real part is the dilaton and whose imaginary part is the axion. This action is a slight generalization of the $SU(4)$ version of $N = 4$ supergravity where

$$f(\psi) = e^{-2\psi} . \quad (10.19)$$

Solutions to the field equations arising from (10.18) and (10.19) have been found by Gibbons and others. These carry central singularities hidden by a horizon. Here we would like to show that again there are no singularity free solutions for a wide class of gravitational fields.

For simplicity ignore the field associated with the magnetic charge and therefore set

$$G_{\mu\lambda} = 0 . \quad (10.20)$$

We adopt the Lagrangian $R + L$ where

$$L = \frac{1}{2}g^{\mu\lambda}\nabla_{(\mu}\psi\nabla_{\lambda)}\psi^* - f_1(\psi)F_{\mu\lambda}F^{\mu\lambda} - f_2(\psi) . \quad (10.21)$$

Here

$$\nabla_\mu = \partial_\mu - ieA_\mu - \Gamma_\mu \quad (10.22)$$

and we have also added a second nonlinear term, which may include a mass term for the scalar field. Here Γ_μ represents the gravitational coupling

One now has the gravitational equation (10.4) and

$$\Theta_{\mu\lambda} = \frac{1}{2}\nabla_{(\mu}\psi\nabla_{\lambda)}^*\psi^* - 2f_1(\psi)\theta_{\mu\lambda}^{e\ell} - \frac{1}{2}f_2(\psi)g_{\mu\lambda} \quad (10.23)$$

where

$$\theta_{\mu\lambda}^{e\ell} = F_{\mu\sigma}(\psi)F_{\lambda}^{\sigma}(\psi) - \frac{1}{4}F_{\alpha\beta}(\psi)F^{\alpha\beta}(\psi)g_{\mu\lambda} . \quad (10.24)$$

The source of $F_{\mu\lambda}(\psi)$ is the charged scalar, ψ :

$$F_{\mu\lambda} = \partial_\mu A_\lambda - \partial_\lambda A_\mu \quad (10.25)$$

$$\square A_\mu = j_\mu(\psi) \quad (10.26)$$

where

$$j_\mu(\psi) \sim \psi^*\nabla_\mu\psi - \psi\nabla_\mu\psi^* . \quad (10.27)$$

The nonlinear equation of motion of the scalar field is

$$g^{\mu\lambda}\nabla_\mu\nabla_\lambda\psi - \frac{\partial f_1(\psi)}{\partial\psi}F_{\mu\lambda}(\psi)F^{\mu\lambda}(\psi) - \frac{\partial f_2(\psi)}{\partial\psi} = 0 . \quad (10.28)$$

If ψ is complex (10.26) and (10.28) are strongly coupled. In addition to (10.26) and (10.28) one has the gravitational equations (10.4).

In the paper of Kallosh *et al.*, where ψ is real, the ansatz for ds is

$$ds^2 = e^{2u} dt^2 - e^{-2u} dr^2 - R^2 d\Omega \quad (10.29)$$

and one finds

$$e^{2u} = \frac{(r - r_-)(r - r_+)}{R^2} \quad (10.30)$$

$$e^{2\psi} = e^{2\psi_o} \frac{r + \Sigma}{r - \Sigma} \quad (10.31)$$

where

$$R^2 = r^2 - \Sigma^2 . \quad (10.32)$$

The curvature singularity is at $r = |\Sigma|$ which is shielded by the horizon. Σ is determined by the mass, charges, and asymptotic value of the dilaton field.

We shall here adopt the Kerr-Schild metric (2.3). This metric is chosen because it is able to accommodate the charged rotating source. Although it also displays a central singularity when the source of mass and charge is confined to a point, it is at least *a priori* possible that the singularity will disappear if the source of mass and charge is spread out as it would be if the charged scalar field is also spread out. We shall investigate this point by examining the gravitational field equations (10.4).

Let us consider the spherically symmetric non-rotating case (corresponding to Reissner-Nordstrom rather than Kerr-Newman). Then the vector potential vanishes.

Assume harmonic time-dependence of ψ :

$$\psi = Re^{i\omega t} . \quad (10.33)$$

Then

$$\nabla_o \psi = i(\omega - eA_o)\psi . \quad (10.34)$$

We may rewrite (10.23)

$$\Theta_{\mu\lambda} = \frac{1}{2} \nabla_{(\mu} \psi \nabla_{\lambda)}^* \psi^* - [a(\psi) \eta_{\mu\lambda} + b(\psi) \ell_\mu \ell_\lambda] \quad (10.35)$$

where $a(\psi)$ and $b(\psi)$ are new scalars determined by $f_1(\psi)$, $f_2(\psi)$, and $\theta_{\mu\lambda}^{el}$. Then, if R is real,

$$\Theta_{oo} = (\omega - eA_o)^2 R^2 - [a(\psi) + b(\psi)\varphi] \quad (10.36)$$

where

$$\varphi = \ell_o^2 \quad (10.37)$$

and

$$\begin{aligned} \Theta_{ok} &= -b(\psi)\varphi\lambda_k \\ \Theta_{jk} &= a(\psi)\delta_{jk} - b(\psi)\varphi\lambda_j\lambda_k + \left(\frac{dR}{dr}\right)^2 \lambda_j\lambda_k . \end{aligned} \quad (10.38)$$

Here

$$\ell_k = \ell_o\lambda_k \quad (10.39)$$

and λ_k is a unit radial vector.

The Ricci tensor is again given by (10.8). Then the gravitational equations become

$$-m\nabla^2\varphi + 2m^2\varphi\nabla^2\varphi = K[\tilde{\omega}^2 R^2 - (a(\psi) + b(\psi)\varphi)] \quad (10.40)$$

$$(2m^2\varphi\nabla^2\varphi)\lambda_k = -Kb(\psi)\varphi\lambda_k \quad (10.41)$$

$$2m\left(\frac{\varphi'}{r} + \frac{\varphi}{r^2}\right) = Ka(\psi) \quad (10.42)$$

$$m\left(\varphi'' - \frac{2\varphi}{r^2}\right) + 2m^2\varphi\nabla^2\varphi = K[(R')^2 - b(\psi)\varphi] \quad (10.43)$$

where

$$\tilde{\omega} = \omega - eA_o . \quad (10.44)$$

By (10.41)

$$\nabla^2\varphi = -\frac{K}{2m^2}b(\psi) . \quad (10.45)$$

By (10.40) and (10.41)

$$\nabla^2\varphi = -\frac{K}{m}[\tilde{\omega}^2 R^2 - a(\psi)] . \quad (10.46)$$

By (10.46) and (10.42)

$$2m\left(\frac{\varphi'}{r} + \frac{\varphi}{r^2}\right) = m\nabla^2\varphi + K\tilde{\omega}^2 R^2 \quad (10.47)$$

or

$$m\left[\frac{2\varphi}{r^2} - \varphi''\right] = K\tilde{\omega}^2 R^2 . \quad (10.48)$$

By (10.43)

$$2m^2\varphi\nabla^2\varphi + Kb(\psi)\varphi = K[(R')^2 + \tilde{\omega}^2 R^2] . \quad (10.49)$$

By (10.45)

$$(R')^2 + \tilde{\omega}^2 R^2 = 0 . \quad (10.50)$$

If R is not real, the argument is unchanged but R is replaced by $|R|$. Since $|R|$ and $\tilde{\omega}$ are real,

$$|R|' = |R| = 0 . \quad (10.51)$$

In additiion, by (10.48)

$$\varphi'' = \frac{2\varphi}{r^2} \quad (10.52)$$

with the independent solutions

$$\varphi \sim r^2 \quad (10.53)$$

and

$$\varphi \sim \frac{1}{r} . \quad (10.54)$$

φ satisfies solitonic boundary conditions at the origin according to (10.53) and at infinity according to (10.54). Finally (10.51) is inconsistent with a non-trivial solution of (10.28). Once again it is not possible to find a genuine soliton.

If there actually is a physical basis for associating elementary particles with singularity free solitons, however, it should not be easy to construct these structures. It would be more reasonable to expect success only with fundamenttnal theories having established physical content.

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